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1979 J. Phys. A: Math. Gen. 12 1441

(http://iopscience.iop.org/0305-4470/12/9/012)

J. Phys. A: Math. Gen., Vol. 12, No. 9, 1979. Printed in Great Britain

Transmission, reflection and diffraction of Love waves in an infinite strip with a surface step

M H Kazi†

Department of Mathematics, Punjab University (New Campus), Lahore, Pakistan

Received 3 January 1978

Abstract. A method based on an integral equation formulation together with the application of the Schwinger–Levine variational principle has been used to investigate the two-dimensional problem of the propagation of plane, harmonic, Love-type waves, incident normally (from either side) upon the step in a structure consisting of an infinite strip with a surface step. Diffraction of Love waves is described exactly by means of a scattering matrix, and approximate expressions for its elements are sought by plane-wave and variational methods. Complex reflection and transmission coefficients may then be obtained through a related transmission matrix.

1. Introduction

The problem of the diffraction of seismic surface waves at continental margins, mountain roots, etc is of considerable importance in geophysics. Unfortunately, the problem defies exact solution. Various authors have idealised a continental margin by a step change in elevation and have employed various analytical and numerical techniques to solve the problem with varying degrees of success (Sato 1961, Knopoff and Hudson 1964, Mal and Knopoff 1965, Alsop 1966, Gregersen and Alsop 1974, 1976). Recently the author (Kazi 1978) has used a method based on an integral representation and the Schwinger–Levine variational principle to investigate the diffraction of plane, harmonic, monochromatic Love waves incident normally upon the plane of discontinuity in a structure consisting of a half-space with a surface step. In this paper we use the same method to consider the same problem when the substratum is replaced by an infinite strip. These finite-depth problems possess certain mathematical and physical characteristics which distinguish them from the corresponding half-space problems. In an earlier paper (Kazi 1975) the author pointed out the difference between the mathematical behaviour of the two-dimensional Love-wave operators governing the propagation of monochromatic SH waves in laterally uniform, layered structures of finite and semi-infinite depths. The spectrum of eigenvalues was shown to be purely discrete in the former case and a disjoint union of the discrete and continuous spectra in the latter. This makes the natures of the two problems very different, and necessitates separate investigation of the finite-depth problem.

In this paper we first describe exactly the diffraction of Love waves through a scattering matrix formulation. Approximate expressions for the elements of the scattering matrix are then obtained through the plane-wave approximation. A general

† Present address: Department of Mathematical Sciences, University of Petroleum and Minerals, Dharan.

formula is derived from the transmission matrix, and some special cases are considered in which the transmission matrix assumes a fairly compact form. We then use the variational principle due to Schwinger and Levine (Moisewitsch 1966) to achieve the variational improvement of the plane-wave approximation in order to incorporate indirectly the effects of non-propagated modes. Explicit formulae for complex reflection and transmission coefficients can be obtained with the help of the transmission matrices.

2. Equations of motion

Consider a surface layer of rigidity μ_1, shear velocity β_1, density ρ_1 and variable thickness h_1, $h_2(>h_1)$ overlying a layer of rigidity $\mu_2(>\mu_1)$, shear velocity $\beta_2(>\beta_1)$, density ρ_2 and thickness $H-h_1$. Both the layers are assumed to be homogeneous and isotropic. Coordinate axes are chosen in such a way that the z axis is vertically downward, the interface is given by $z=h_1$, and the step in the surface of the upper layer is taken to lie in the plane $x=0$ (see figure 1). The thickness of the upper layer is taken to be h_1 for $x<0$ and h_2 for $x>0$, and we write $\delta=h_2-h_1$.

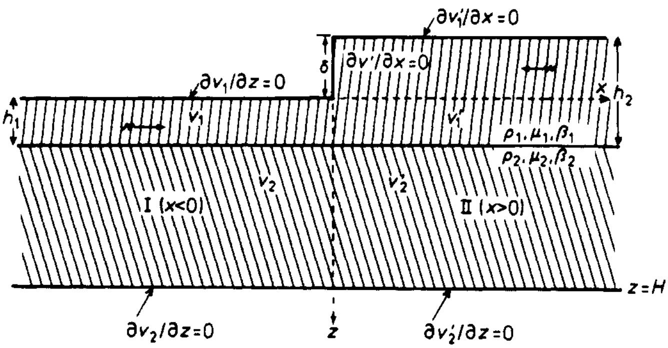

Figure 1.

We consider only the two-dimensional problems of the propagation of Love waves incident normally (from either side) upon the step, and postulate the time dependence $\exp(-i\omega t)$, ω being the angular frequency. Thus the wave motion is entirely SH in character. The displacement fields in domains I($x<0$) and II($x>0$) are denoted by

$$\exp(-i\omega t)v(x,z)=\begin{cases}\exp(-i\omega t)v_1(x,z), & 0\leqslant z\leqslant h_1, \quad x<0\\ \exp(-i\omega t)v_2(x,z), & h_1\leqslant z\leqslant H, \quad x<0\end{cases} \tag{1}$$

and

$$\exp(-i\omega t)v'(x,z)=\begin{cases}\exp(-i\omega t)v'(x,z), & -\delta\leqslant z\leqslant h_1, \quad x>0\\ \exp(-i\omega t)v_2^1(x,z), & h_1\leqslant z\leqslant H, \quad x>0\end{cases} \tag{2}$$

respectively.

The surfaces $z=0$, $x<0$, $z=-\delta=h_1-h_2$, $x>0$, $z=H$ $\forall x$ and the vertical surface of the step are stress-free. Thus

$$\partial v_1/\partial z=0 \qquad \text{at } z=0, \qquad x<0, \tag{3a}$$

$$\partial v_1'/\partial z = 0 \qquad \text{at } z = -\delta, \qquad x > 0, \tag{3b}$$

$$\partial v_2/\partial z = 0, \qquad (x < 0), \qquad \partial v_2'/\partial z = 0, \qquad (x > 0) \qquad \text{at } z = H \tag{3c}$$

and

$$\partial v_1'/\partial x = 0 \qquad \text{at } x = 0, \qquad -\delta \leqslant z < 0. \tag{3d}$$

The complete solution for the displacement $v(x, z)$ in domain $\mathrm{I}(x < 0)$ can be expressed in terms of the complete set of eigenfunctions (found in Kazi 1976) associated with the Love-wave operator for an infinite strip consisting of a layer of depth $H - h_1$, rigidity μ_2 and shear velocity β_2, overlain by a layer of depth h_1, rigidity μ_1 and shear velocity β_1. Likewise, the complete solution for the displacement $v'(x, z)$ in domain $\mathrm{II}(x > 0)$ can be expressed in terms of the eigenfunctions for an infinite strip consisting of a layer of depth $H - h_1$, rigidity μ_2 and shear velocity β_2, overlain by a layer of depth h_2, rigidity μ_1 and shear velocity β_1. Thus in domain I

$$v(x, z) = -\left(\sum_{m=1}^{r} (A_m \exp(-\mathrm{i}k_m|x|) + B_m \exp(\mathrm{i}k_m|x|)\chi_m(z) + \sum_{j=1}^{\infty} C(k_j) \exp(-k_j|x|)\, \psi(z, k)\right), \qquad x < 0 \tag{4}$$

and in domain II

$$v'(x, z) = \sum_{m=1}^{s} [A'_m \exp(-\mathrm{i}k'_m x) + B'_m \exp(\mathrm{i}k'_m x)]\chi'_m(z) + \sum_{j=1}^{\infty} C'(k'_j) \exp(-k'_j x)\, \psi'(z, k'_j), \qquad x > 0, \tag{5}$$

where (as in Kazi 1976)

$$\chi_m(z) = \begin{cases} \phi_1^{(m)}(z), & 0 \leqslant z \leqslant h_1 \\ \phi_2^{(m)}(z), & h_1 \leqslant z \leqslant H, \end{cases} \tag{6}$$

$$\chi'_m(z) = \begin{cases} \phi_1'^{(m)}(z), & -\delta \leqslant z \leqslant h_1 \\ \phi_2'^{(m)}(z), & h_1 \leqslant z \leqslant H, \end{cases} \tag{7}$$

$$\phi_{(z)}^{(m)} = D_m \cos(\sigma_1^{(m)} z)/\cos(\sigma_1^{(m)} h_1), \tag{8}$$

$$\phi_2'^{(m)}(z) = D_m \cosh[\sigma_2^{(m)}(z - H)]/\cosh[\sigma_2^{(m)}(H - h_1)], \tag{9}$$

$$D_m = 2\left(\frac{\sigma_2^{(m)}}{\mu_2}\right)^{1/2} \left(\frac{\beta_1^{-2} - U_m^{-1} C_m^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right)^{1/2} \frac{\cosh[\sigma_2^{(m)}(H - h_1)]}{\{\sinh[2\sigma_2^{(m)}(H - h_1)] + 2\sigma_2^{(m)}(H - h_1)\}^{1/2}}, \tag{10}$$

$$\phi_1'^{(m)}(z) = D'_m \cos[\sigma_1'^{(m)}(z + \delta)]/\cos(\sigma_1'^{(m)} h_2), \tag{11}$$

$$\phi_2'^{(m)}(z) = D'_m \cosh[\sigma_2'^{(m)}(z - H)]/\cosh[\sigma_2'^{(m)}(H - h_1)], \tag{12}$$

$$D'_m = 2\left(\frac{\sigma_2'^{(m)}}{\mu_2}\right)^{1/2} \left(\frac{\beta_1^{-2} - U_m'^{-1} C_m'^{-1}}{\beta_1^{-2} - \beta_2^{-2}}\right)^{1/2} \frac{\cosh[\sigma_2'^{(m)}(H - h_1)]}{\{\sinh[2\sigma_2'^{(m)}(H - h_1)] + 2\sigma_2'^{(m)}(H - h_1)\}^{1/2}}. \tag{13}$$

U_m, U'_m denote the group velocities, C_m, C'_m the phase velocities in the two mth modes,

$$\sigma_1(\lambda) = (\omega^2/\beta_1^2 - \lambda)^{1/2}, \qquad \sigma_2(\lambda) = (\lambda - \omega^2/\beta_2^2)^{1/2}, \qquad \lambda = k^2, \tag{14}$$

$$\sigma_1^{(m)} = \sigma_1(\lambda_m), \qquad \sigma_2^{(m)} = \sigma_2(\lambda_m), \tag{15}$$

where $\lambda_m = k_m^2$, $k_m > 0$, $m = 1, 2, \ldots, r$, λ_m being the r positive real eigenvalues satisfying the dispersion equation

$$\mu_1\sigma_1 \tan \sigma_1 h_1 - \mu_2\sigma_2 \tanh \sigma_2(H - h_1) = 0. \tag{16}$$

In $x > 0$,

$$\sigma_1'(\lambda') = (\omega^2/\beta_1^2 - \lambda')^{1/2}, \qquad \sigma_2'(\lambda') = (\lambda' - \omega^2/\beta_2^2)^{1/2}, \qquad \lambda' = k'^2, \tag{17}$$

$$\sigma_1'^{(m)} = \sigma_1(\lambda_m'), \qquad \sigma_2'^{(m)} = \sigma_2(\lambda_m'), \tag{18}$$

$$\lambda_m' = k_m'^2, \qquad k_m' > 0, \qquad m = 1, 2, \ldots, s, \tag{19}$$

λ_m' being the s real positive eigenvalues satisfying the period equation

$$\mu_1\sigma_1' \tan \sigma_1' h_2 - \mu_2\sigma_2' \tanh \sigma_2'(H - h_1) = 0. \tag{20}$$

In addition to the aforementioned roots, the eigenvalue equations (16) and (20) have infinite, discrete sets $\{\lambda_j\}$, $\{\lambda_j'\}$ of negative real roots respectively: $\lambda_j = (\mathrm{i}k_j)^2$ and $\lambda_j' = (\mathrm{i}k_j')^2$, $j = 1, 2, \ldots, k_j$ and k_j' being real and positive. The eigenfunctions corresponding to these eigenvalues are given by

$$\psi(z, k_j) = \begin{cases} \psi_1(z, k_j), & 0 \leqslant z \leqslant h_1 \\ \psi_2(z, k_j), & h_1 \leqslant z \leqslant H \end{cases} \tag{21}$$

and

$$\psi'(z, k_j') = \begin{cases} \psi_1'(z, k_j'), & -\delta \leqslant z \leqslant h_1 \\ \psi_2'(z, k_j'), & h_1 \leqslant z \leqslant H \end{cases}, \qquad j = 1, 2, \ldots, \tag{22}$$

where $\psi_1(z, k_j)$ and $\psi_2(z, k_j)$ have expressions similar to (8) and (9) with $\sigma_1^{(m)}$, $\sigma_2^{(m)}$ replaced by $\sigma_1(\lambda_j) = \sigma_1(-k_j^2)$ and $\sigma_2(\lambda_j) = \sigma_2(-k_j^2)$ respectively. The functions $\psi_1'(z, k_j')$ and $\psi_2'(z, k_j')$ are obtained similarly.

The eigenfunctions listed above satisfy the orthonormality relations

$$\int_0^H \mu(z)\chi_m(z)\chi_n(z)\,\mathrm{d}z = \delta_{mn}, \qquad 1 \leqslant m,n \leqslant r, \tag{23a}$$

$$\int_{-\delta}^H \mu(z)\chi_m'(z)\chi_n'(z)\,\mathrm{d}z = \delta_{mn}, \qquad 1 \leqslant m,n \leqslant s, \tag{23b}$$

$$\int_0^H \mu(z)\chi_m(z)\psi(z, k_j)\,\mathrm{d}z = 0, \qquad 1 \leqslant m \leqslant r, \qquad 1 \leqslant j, \tag{23c}$$

$$\int_{-\delta}^H \mu(z)\chi_m'(z)\psi'(z, k_j')\,\mathrm{d}z = 0, \qquad 1 \leqslant m \leqslant s, \qquad 1 \leqslant j, \tag{23d}$$

$$\int_0^H \mu(z)\psi(z, k_i)\psi(z, k_j)\,\mathrm{d}z = \delta_{ij}, \qquad i, j \geqslant 1, \tag{23e}$$

$$\int_{-\delta}^H \mu(z)\psi'(z, k_i')\psi'(z, k_j')\,\mathrm{d}z = \delta_{ij}, \qquad i, j \geqslant 1, \tag{23f}$$

where

$$\mu(z) = \begin{cases} \mu_1, & -\delta \leqslant z \leqslant h_1 \text{ in } x > 0 \text{ and } 0 \leqslant z \leqslant h_1 \text{ in } x < 0, \\ \mu_2, & h_1 < z \leqslant H. \end{cases} \tag{24}$$

3. Integral equation formulation

Let $\tau(z)$ denote the component τ_{xy} of stress at any point in the plane $x=0$:

$$\tau(z)=\tau_{xy}|_{x=0}=\mu(z)\partial v/\partial x|_{x=0^-}=\mu(z)\partial v'/\partial x|_{x=0^+}, \qquad z\geq 0. \tag{25}$$

Equation (3d) implies that $\tau(z)=0$ when $-\delta\leq z<0$.

In domain I

$$\tau(z)=\mu(z)\frac{\partial v}{\partial x}\bigg|_{x\to 0^-}=-\mu(z)\Big(\sum_{m=1}^{r}\mathrm{i}k_m(A_m-B_m)\chi_m(z)+\sum_{j=1}^{\infty}k_jC(k_j)\psi(z,k_j)\Big) \tag{26}$$

and in domain II

$$\tau(z)=\mu(z)\frac{\partial v'}{\partial x}\bigg|_{x\to 0^+}=-\mu(z)\Big(\sum_{m=1}^{s}\mathrm{i}k'_m(A'_m-B'_m)\chi'_m(z)+\sum_{j=1}^{\infty}k'_jC'(k'_j)\psi'(z,k'_j)\Big). \tag{27}$$

Multiplying equation (26) by $\chi_m(z)$, $m=1,2,\ldots,r$ and $\psi(z,k_j)$, $j\geq 1$, and integrating both sides with respect to z over the interval $(0,H)$, we obtain

$$-\mathrm{i}k_m(A_m-B_m)=\int_0^H\tau(\eta)\chi_m(\eta)\,\mathrm{d}\eta, \qquad m=1,2,\ldots,r, \tag{28}$$

$$-k_jC(k_j)=\int_0^H\tau(\eta)\psi(\eta,k_j)\,\mathrm{d}\eta, \qquad j=1,2,\ldots. \tag{29}$$

Likewise, if we multiply equation (27) in succession by $\chi'_m(z)$, $m=1,2,\ldots,s$, $\psi'(z,k'_j)$, $j=1,2,\ldots$ and integrate with respect to z over the interval $(-\delta,H)$, we obtain

$$-\mathrm{i}k'_m(A'_m-B'_m)=\int_{-\delta}^H\tau(\eta)\chi'_m(\eta)\,\mathrm{d}\eta=\int_0^H\tau(\eta)\chi'_m(\eta)\,\mathrm{d}\eta, \qquad m=1,2,\ldots,r \tag{30}$$

($\because\tau(\eta)=0$ in the interval $[-\delta,0]$),

$$-k'_jC'(k'_j)=\int_{-\delta}^H\tau(\eta)\psi'(\eta,k'_j)\,\mathrm{d}\eta=\int_0^H\tau(\eta)\psi'(\eta,k'_j)\,\mathrm{d}\eta, \qquad j=1,2,\ldots. \tag{31}$$

Substituting the expressions for $C'(k'_j)$ and $C(k_j)$ from (31) and (29) in (4) and (5) and invoking the matching condition $v(0,z)=v'(0,z)$, $0\leq z\leq H$, we obtain

$$\sum_{m=1}^{r}(A_m+B_m)\chi_m(z)+\sum_{m=1}^{s}(A'_m+B'_m)\chi'_m(z)=\int_0^H G(z,\eta)\tau(\eta)\,\mathrm{d}\eta, \qquad 0<z\leq H, \tag{32}$$

where

$$G(z,\eta)=\sum_{j=1}^{\infty}\Big(\frac{\psi(z,k_j)\psi(\eta,k_j)}{k_j}+\frac{\psi'(z,k'_j)\psi'(\eta,k'_j)}{k'_j}\Big). \tag{33}$$

We emphasise that (32) is valid only for the interval $0<z\leq H$ (and not for $-\delta\leq z<0$). Equations (28), (30), (32) and (33) constitute the integral equation formulation of the problem. Given the amplitudes of the incident modes (i.e. A_m's and A'_m's), we have to find the amplitudes of the transmitted and reflected modes (i.e. B_m's and B'_m's) from the $r+s+1$ equations (28), (30) and (32) in B's and $\tau(\eta)$. We note that $\tau(\eta)$ is unknown and so direct solution for B's is not possible. Ultimately we will introduce plausible estimates in $\tau(\eta)$ and thence obtain estimates of B's. We find it convenient to restate the problem in matrix form.

4. Matrix formulation

Let

$$\boldsymbol{A}=\begin{pmatrix}A_1\\A_2\\\vdots\\A_r\\A'_1\\\vdots\\A'_s\end{pmatrix},\qquad \boldsymbol{B}=\begin{pmatrix}B_1\\B_2\\\vdots\\B_r\\B'_1\\\vdots\\B'_s\end{pmatrix}\qquad\text{and}\qquad \boldsymbol{\chi}(z)=\begin{pmatrix}\chi_1(z)\\\vdots\\\chi_r(z)\\\chi'_1(z)\\\vdots\\\chi'_s(z)\end{pmatrix}\tag{34}$$

be $n\times 1$ matrices, $n=r+s$ and

$$\boldsymbol{K}=\begin{pmatrix}k_1 & & & & & & \\ & k_2 & & & & \mathbf{0} & \\ & & \ddots & & & & \\ & & & k_r & & & \\ & & & & k'_1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & k'_s\end{pmatrix}\tag{35}$$

be an $n\times n$ diagonal matrix. Then we can rewrite equations (28), (30) and (32) in matrix notation as follows:

$$-\mathrm{i}\boldsymbol{K}\,.\,(\boldsymbol{A}-\boldsymbol{B})=\int_0^H \boldsymbol{\chi}(\eta)\tau(\eta)\,\mathrm{d}\eta,\tag{36}$$

$$(\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}^{\mathrm{T}})\,.\,\boldsymbol{\chi}(z)=\int_0^H G(z,\eta)\tau(\eta)\,\mathrm{d}\eta,\tag{37}$$

where the superscript T denotes the transpose.

These equations can be considered in the following way: if the amplitudes of the propagated modes in the combinations A_m+B_m, $m=1,2,\ldots,r$, $A'_m+B'_m$, $m=1,2,\ldots,s$ are given, we may find $\tau(\eta)$ to satisfy (37) and then determine A_m-B_m, $m=1,2,\ldots,r$, $A'_m-B'_m$, $m=1,2,\ldots,s$ from (36). This effectively yields $n=r+s$ linear relations amongst the $2(r+s)$ coefficients $A_1,\ldots,A_r$, $B_1,\ldots,B_r$, $A'_1,\ldots,A'_s$ and $B'_1,\ldots,B'_s$.

From equations (36) and (37), we see that (i) $\boldsymbol{A}-\boldsymbol{B}$ and $\boldsymbol{A}+\boldsymbol{B}$ must be linearly related, and (ii) the unknown stress $\tau(z)$ must be linear in $\boldsymbol{A}+\boldsymbol{B}$.

As a consequence of (i), there exists an $n\times n$ matrix $\boldsymbol{S}$ such that

$$\boldsymbol{K}\,.\,(\boldsymbol{A}-\boldsymbol{B})=\mathrm{i}\boldsymbol{S}\,.\,(\boldsymbol{A}+\boldsymbol{B}).\tag{38}$$

The $n\times n$ matrix $\boldsymbol{S}=\|s_{ij}\|$ is called the *scattering matrix*.

Rearranging (38), we obtain

$$(\boldsymbol{K}-\mathrm{i}\boldsymbol{S})\,.\,\boldsymbol{A}=(\boldsymbol{K}+\mathrm{i}\boldsymbol{S})\,.\,\boldsymbol{B}\tag{39}$$

or

$$\boldsymbol{B}=\boldsymbol{T}\,.\,\boldsymbol{A},\tag{40}$$

where

$$\boldsymbol{T}=(\boldsymbol{K}+\mathrm{i}\boldsymbol{S})^{-1}\,.\,(\boldsymbol{K}-\mathrm{i}\boldsymbol{S}),\tag{41}$$

provided $\boldsymbol{K}+\mathrm{i}\boldsymbol{S}$ is non-singular.

As a consequence of (ii), we can introduce an $n \times 1$ matrix

$$\boldsymbol{\tau}(z) = \begin{pmatrix} \tau_1(z) \\ \vdots \\ \tau_r(z) \\ \tau_1'(z) \\ \vdots \\ \tau_s'(z) \end{pmatrix} \tag{42}$$

such that

$$\tau(z) = (\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}) \,.\, \boldsymbol{\tau}(z). \tag{43}$$

Substituting (43) into (37), we obtain

$$(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}) \,.\, \left(\boldsymbol{\chi}(z) - \int_0^H G(z, \eta)\boldsymbol{\tau}(\eta)\,\mathrm{d}\eta\right) = 0,$$

whence

$$\boldsymbol{\chi}(z) = \int_0^H G(z, \eta)\boldsymbol{\tau}(\eta)\,\mathrm{d}\eta, \tag{44}$$

on account of the linear independence and arbitrary character of $A_1 + B_1, \ldots, A_r + B_r, A_1' + B_1', \ldots, A_s' + B_s'$.

Equation (44) yields n uncoupled integral equations

$$\chi_m(z) = \int_0^H G(z, \eta)\tau_m(\eta)\,\mathrm{d}\eta, \qquad m = 1, 2, \ldots, r \tag{45}$$

and

$$\chi_m'(z) = \int_0^H G(z, \eta)\tau_m'(\eta)\,\mathrm{d}\eta, \qquad m = 1, 2, \ldots, s \tag{46}$$

for the determination of $\boldsymbol{\tau}(z)$.

Substituting for $\tau(\eta)$ from equation (43) in (36), we obtain

$$\boldsymbol{K} \,.\, (\boldsymbol{A} - \boldsymbol{B}) = \mathrm{i} \int_0^H \boldsymbol{\chi}(\eta) \,.\, [(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}) \,.\, \boldsymbol{\tau}(\eta)]\,\mathrm{d}\eta,$$

whence on replacing $\boldsymbol{K} \,.\, (\boldsymbol{A} - \boldsymbol{B})$ by $\mathrm{i}\boldsymbol{S} \,.\, (\boldsymbol{A} + \boldsymbol{B})$ (from equation (38)), we obtain

$$\boldsymbol{S} \,.\, (\boldsymbol{A} + \boldsymbol{B}) = \int_0^H \boldsymbol{\chi}(\eta) \,.\, [(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}) \,.\, \boldsymbol{\tau}(\eta)]\,\mathrm{d}\eta$$

or

$$s_{ij} = \int_0^H \chi_i(\eta)\tau_j(\eta)\,\mathrm{d}\eta, \qquad i, j = 1, 2, \ldots, r, r+1, \qquad n = r + s, \tag{47}$$

where

$$\chi_{r+t} = \chi_t' \qquad \text{and} \qquad \tau_{r+t} = \tau_t', \qquad t = 1, 2, \ldots, s.$$

Our problem is now reduced to the solution of the integral equations (45) and (46) together with the determination of the scattering matrix from (47). The matrix $\boldsymbol{T}$

(called the *transmission matrix*) in (40) or (41) will then yield the complex reflection and transmission coefficients. We define the complex reflection (transmission) coefficients as the ratios of the (complex) amplitudes of reflected (transmitted) and incident waves.

So far, the formulation of the problem is exact. The exact solution is, however, not possible in practice, and we must seek approximate expressions for the elements of the scattering matrix. We may, for instance, return to § 3 to construct a plane-wave approximation (see § 5), or we may construct expressions for the matrix elements s_{ij} to which a variational principle applies and can be used to improve the approximation (see § 6).

5. Plane-wave approximation

The plane-wave approximation consists of the neglect of non-propagated modes $\psi(z, k_j)$, $\psi'(z, k_j')$, $j = 1, 2, \ldots$ and therefore of the kernel $G(z, \eta)$ in the formulation of § 3. Thus we may omit the contribution to τ from non-propagated modes and expand τ in terms of $\chi_m (m = 1, 2, \ldots, r)$ or $\chi'_m (m = 1, 2, \ldots, s)$. We choose the former to obtain

$$\tau(z) = \mu(z) \sum_{m=1}^{r} D_m \chi_m(z) \tag{48}$$

and rewrite equation (36) in the form

$$-\mathrm{i}\boldsymbol{K}.(\boldsymbol{A} - \boldsymbol{B}) = \int_0^H \begin{pmatrix} \chi_1(\eta) \\ \vdots \\ \chi_r(\eta) \\ \chi_1'(\eta) \\ \vdots \\ \chi_s'(\eta) \end{pmatrix} \mu(\eta) \left(\sum_{m=1}^{r} D_m \chi_m(\eta) \right) \mathrm{d}\eta \tag{49}$$

or equivalently (on using orthonormality)

$$-\mathrm{i} \begin{pmatrix} k_1(A_1 - B_1) \\ \vdots \\ k_r(A_r - B_r) \\ k_1'(A_1' - B_1') \\ \vdots \\ k_s'(A_s' - B_s') \end{pmatrix} \begin{pmatrix} D_1 \\ \vdots \\ D_r \\ \sum_{m=1}^{r} P_{1m}\lambda_{1m}D_m \\ \vdots \\ \sum_{m=1}^{r} P_{sm}\lambda_{sm}D_m \end{pmatrix}, \tag{50}$$

where the *coupling coefficients* P_{im} are given by the integrals

$$\lambda_{im}P_{im} = \int_0^H \mu(\eta)\chi_i'(\eta)\chi_m(\eta)\,\mathrm{d}\eta, \qquad i = 1, 2, \ldots, s, \qquad m = 1, 2, \ldots, r, \tag{51}$$

with

$$\lambda_{im} = (k_i'/k_m)^{1/2}. \tag{52}$$

The factor λ_{im} has been introduced in (51) for the sake of convenience. After some effort, we obtain

$$\lambda_{im}P_{im}=\mu_1\bar{D}_i'\bar{D}_m\sigma_1^{(i)\prime}\sin(\sigma_1^{(i)\prime}\delta)/(k_i'^2-k_m^2)\cos(\sigma_1^{(m)}h_1)\cos(\sigma_1^{(i)\prime}h_2), \tag{53}$$

where D_i', D_m are given by the RHS of (13) and (10) respectively, and

$$k_m^2-k_i'^2=(\sigma_1^{(i)\prime})^2-(\sigma_1^{(m)})^2=(\sigma_2^{(m)})^2-(\sigma_2^{(i)\prime})^2. \tag{54}$$

Eliminating $D_1, D_2, \ldots, D_r$ from (50) and simplifying, we obtain

$$\boldsymbol{R}\,.\,(\boldsymbol{A}-\boldsymbol{B})=0, \tag{55}$$

where the $s\times n$ matrix $\boldsymbol{R}$ is given by

$$\boldsymbol{R}=\left(\underbrace{\begin{matrix}P_{11}/\lambda_{11} & P_{12}/\lambda_{12} & \ldots & P_{1r}/\lambda_{1r}\\ P_{21}/\lambda_{21} & P_{22}/\lambda_{22} & \ldots & P_{2r}/\lambda_{2r}\\ \vdots & & & \\ P_{s1}/\lambda_{s1} & P_{s2}/\lambda_{s2} & \ldots & P_{sr}/\lambda_{sr}\end{matrix}}_{s\times r}\;\underbrace{\begin{matrix}-1 & & & \\ & -1 & & \mathbf{0}\\ & \mathbf{0} & \ddots & \\ & & & -1\end{matrix}}_{s\times s}\right). \tag{56}$$

Setting $G=0$ in (37), we obtain

$$(\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{B}^{\mathrm{T}})\,.\,\boldsymbol{\chi}(z)=0. \tag{57}$$

Calculating the first moments of (57) with respect to $\mu(z)\chi_i(z)$, $i=1,2,\ldots,r$, we obtain a set of simultaneous, linear, algebraic equations equivalent to the matrix equation

$$\boldsymbol{Q}\,.\,(\boldsymbol{A}+\boldsymbol{B})=0, \tag{58}$$

where the $r\times n$ matrix $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q}=\begin{pmatrix}1 & & \mathbf{0} & \lambda_{11}P_{11} & \lambda_{21}P_{21} & \ldots & \lambda_{s1}P_{s1}\\ & 1 & & \lambda_{12}P_{12} & \lambda_{22}P_{22} & \ldots & \lambda_{s2}P_{s2}\\ & & \ddots & \vdots & & & \\ \mathbf{0} & & 1 & \lambda_{1r}P_{1r} & \lambda_{2r}P_{2r} & \ldots & \lambda_{sr}P_{sr}\end{pmatrix}. \tag{59}$$

Combining (55) and (58) into a single matrix equation, we obtain

$$\begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{R}\end{pmatrix}.\,\boldsymbol{A}=\begin{pmatrix}-\boldsymbol{Q}\\ \boldsymbol{R}\end{pmatrix}.\,\boldsymbol{B} \tag{60}$$

or

$$\boldsymbol{B}=\boldsymbol{T}\,.\,\boldsymbol{A}\qquad\text{where } \boldsymbol{T}=\begin{pmatrix}-\boldsymbol{Q}\\ \boldsymbol{R}\end{pmatrix}^{-1}.\begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{R}\end{pmatrix}. \tag{61}$$

The matrix $\boldsymbol{T}$ gives the reflection and transmission coefficients.

It is clear from the form of the matrix $\binom{-\boldsymbol{Q}}{\boldsymbol{R}}$ that in the general case $\binom{-\boldsymbol{Q}}{\boldsymbol{R}}^{-1}$ has a very complex analytical expression. However, in some special cases we can find $\binom{-\boldsymbol{Q}}{\boldsymbol{R}}^{-1}$ and the matrix $\boldsymbol{T}$ in a fairly compact form.

(i) When $r=1$, $s \geqslant 1$ (i.e. there is a single (fundamental mode) in the left-hand domain and there are s modes on the right) we obtain after considerable effort

$$\boldsymbol{T}=\frac{1}{N}\begin{pmatrix} -2+N & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} & \dots & -2\lambda_{s1}P_{s1} \\ -2P_{11}/\lambda_{11} & -2P_{11}^2+N & -2P_{11}\lambda_{21}P_{21}/\lambda_{11} & \dots & -2P_{11}\lambda_{s1}P_{s1}/\lambda_{11} \\ -2P_{21}/\lambda_{21} & -2P_{21}P_{11}\lambda_{11}/\lambda_{21} & -2P_{21}^2+N & \dots & -2P_{21}\lambda_{s1}P_{s1}/\lambda_{21} \\ \vdots & & & \ddots & \\ -2P_{s1}/\lambda_{s1} & -2P_{s1}P_{11}\lambda_{11}/\lambda_{s1} & \dots & \dots & -2P_{s1}^2+N \end{pmatrix}, \tag{62}$$

where

$$N=1+P_{11}^2+P_{21}^2+\ldots+P_{sr}^2. \tag{63}$$

(ii) When $r \geqslant 1$, $s=1$ (in this case there is a single fundamental mode in the right-hand domain and there are r modes on the left) we find

$$\boldsymbol{T}=\frac{1}{N}\begin{pmatrix} -N+2P_{11}^2 & 2P_{11}\lambda_{11}P_{12}/\lambda_{12} & \dots & 2P_{11}\lambda_{11}P_{1r}/\lambda_{1r} & -2P_{11}\lambda_{11} \\ 2\lambda_{12}P_{12}P_{11}/\lambda_{11} & -N+2P_{12}^2 & \dots & 2\lambda_{12}P_{12}P_{1r}/\lambda_{1r} & -2P_{12}\lambda_{12} \\ \vdots & & & & \\ 2P_{1r}\lambda_{1r}P_{11}/\lambda_{11} & 2P_{1r}P_{12}/\lambda_{12} & \dots & -N+2P_{1r}^2 & -2P_{1r}\lambda_{1r} \\ -2P_{11}/\lambda_{11} & -2P_{12}/\lambda_{12} & \dots & -N+2 & \end{pmatrix}, \tag{64}$$

where

$$N=1+P_{11}^2+P_{12}^2+\ldots+P_{1r}^2. \tag{65}$$

6. Variational formulation

We now use the exact matrix formulation of the problem in § 4 to construct expressions for the elements of the scattering matrix to which the variational principle of Schwinger and Levine applies.

Multiplying the equation

$$\chi_i(z)=\int_0^H G(z,\eta)\tau_i(\eta)\,\mathrm{d}\eta, \qquad i=1,2,\ldots,r,r+1,\ldots,n=r+s \tag{44}$$

by $\tau_j(z)$ and integrating with respect to z over the interval $(0,H)$, we obtain

$$\int_0^H \chi_i(\eta)\tau_j(\eta)\,\mathrm{d}\eta=\int_0^H\int_0^H \tau_i(z)G(z,\eta)\tau_j(\eta)\,\mathrm{d}\eta, \qquad i,j=1,2,\ldots, \tag{66}$$

and equation (47) gives

$$s_{ij}=\int_0^H\int_0^H G(z,\eta)\tau_i(z)\tau_j(\eta)\,\mathrm{d}z\,\mathrm{d}\eta. \tag{67}$$

Since $G(z, \eta) = G(\eta, z)$, it follows from (67) that $s_{ij} = s_{ji}$, and so the scattering matrix $\boldsymbol{S} = \|s_{ij}\|$ is symmetric. Using (47) and (67) we can write

$$s_{ij} = \int_0^H \chi_i(z)\tau_j(z)\,\mathrm{d}z \int_0^H \chi_j(\eta)\tau_i(\eta)\,\mathrm{d}\eta \Big/ \int_0^H \int_0^H \tau_i(z)G(z, \eta)\tau_j(\eta)\,\mathrm{d}z\,\mathrm{d}\eta. \tag{68}$$

If we introduce the notations $\langle f, u\rangle = \int_0^H fu\,\mathrm{d}z$, $Gu = \int_0^H G(z, \eta)u(\eta)\,\mathrm{d}\eta$, we can rewrite (68) in the form

$$s_{ij} = \langle \chi_i, \tau_j\rangle\langle \chi_j, \tau_i\rangle/\langle G\tau_i, \tau_j\rangle, \qquad i, j = 1, 2, \ldots, n. \tag{69}$$

Using the method given in Stakgold (1967) we can prove the following:

Theorem. Let

$$F(u, v) = \langle \chi_i, v\rangle + \langle \chi_j, u\rangle - \langle Gu, v\rangle.$$

Then F is stationary for variations of u, v about $u = \tau_i$, $v = \tau_j$, where τ_i, τ_j are the solutions of the integral equations

$$\begin{aligned} \chi_i(z) &= \int_0^H G(z, \eta)\tau_i(\eta)\,\mathrm{d}\eta = G\tau_i, \\ \chi_j(z) &= \int_0^H G(z, \eta)\tau_j(\eta)\,\mathrm{d}\eta = G\tau_j \qquad (i, j \text{ fixed}) \end{aligned} \tag{70}$$

respectively. Moreover the stationary value of F is $s_{ij}(\tau_i, \tau_j)$.

Corollary (*Schwinger–Levine variational principle*). Let

$$R(u, v) = \langle \chi_j, u\rangle\langle \chi_i, v\rangle/\langle Gu, v\rangle.$$

Then R is stationary for variations of u, v about $u = \alpha\tau_i$, $v = \beta\tau_j$, where α, β are arbitrary non-zero constants. Moreover,

$$R(\alpha\tau_i, \beta\tau_j) = s_{ij}(\tau_i, \tau_j).$$

It may be remarked that the Schwinger–Levine variational principle ensures the scale independence of the trial functions for τ_i, τ_j.

The variational formulation of the problem enables us to proceed from a first approximation to approximations of an improved accuracy. In order to estimate a certain s_{ij} we may substitute for u an approximation to the solution τ_i of the first of equations (70), and for v an approximation to the solution of the second of equations (70). If the errors in these approximations are $\mathrm{O}(\epsilon)$, the error in the approximation for s_{ij} is $\mathrm{O}(\epsilon^2)$. Alternatively, we might use eigenfunction expansions for τ_i, τ_j in terms of $\chi_m(z)$ and $\psi(z, k_j)$ and then invoke the variational principle to determine the coefficients in the expansions by setting the derivative of R with respect to each coefficient equal to zero. However, the resulting sets of algebraic equations are infinite in number. We choose an approach which is halfway between these two. We assume an expansion of τ_i in terms of the propagated modes only, as for the plane-wave approximation:

$$\tau_i(z) = \sum_{p=1}^{r} D_{ip}\mu(z)\chi_p(z), \qquad i = 1, 2, \ldots, n. \tag{71}$$

We determine the coefficients D_{ip} in the same way as if τ_i were given in terms of the complete set of eigenfunctions. We differentiate R with respect to each D_{ip} and put the result equal to zero. This is a heuristic approach; R may not be stationary in the D_{ip}'s if (71) is not an expansion in terms of a complete set. However, it appears to lead to satisfactory results in certain cases (see e.g. Miles 1946, 1967).

Substituting (71) into the expression for $F(\tau_i, \tau_j)$, we obtain

$$F(\tau_i, \tau_j) = \left\langle \chi_j, \sum_{p=1}^{r} D_{ip}\mu(z)\chi_p(z) \right\rangle + \left\langle \chi_i, \sum_{q=1}^{r} D_{jq}\mu(z)\chi_q(z) \right\rangle$$

$$- \left\langle \int_0^H G(z, \eta) \sum_{p=1}^{r} D_{ip}\mu(\eta)\chi_p(\eta)\,\mathrm{d}\eta, \sum_{q=1}^{r} D_{pq}\mu(z)\chi_q(z) \right\rangle$$

$$= \sum_{p=1}^{r} D_{ip}\langle \chi_j, \mu(z)\chi_p(z)\rangle + \sum_{q=1}^{r} D_{jq}\langle \chi_i, \mu(z)\chi_q(z)\rangle - \sum_{q=1}^{r}\sum_{p=1}^{r} D_{ip}D_{jq}I_{pq}, \tag{72}$$

where

$$I_{pq} = \int_0^H \left(\int_0^H G(z, \eta)\chi_p(\eta)\mu(\eta)\,\mathrm{d}\eta \right) \mu(z)\chi_q(z)\,\mathrm{d}z \tag{73}$$

and

$$G(z, \eta) = \sum_{j=1}^{\infty} \left(\frac{\psi(z, k_j)\psi(\eta, k_j)}{k_j} + \frac{\psi'(\eta, k_j')\psi'(z, k_j')}{k_j'} \right)$$

(cf equation (33)).

Substituting for $G(z, \eta)$ in equation (73) and using the orthogonality relations

$$\int_0^H \chi_p(z)\psi(z, k_j)\mu(z)\,\mathrm{d}z = 0, \qquad p = 1, 2, \ldots, r, j \geqslant 1,$$

we obtain

$$I_{pq} = \sum_j \frac{1}{k_j} \int_0^H \int_0^H \mu(\eta)\chi_p(\eta)\psi'(z, k_j')\psi'(\eta, k_j')\mu(z)\chi_q(z)\,\mathrm{d}z\,\mathrm{d}\eta$$

$$= \sum_j \frac{1}{k_j'} \int_0^H \mu(\eta)\psi'(\eta, k_j')\chi_p(\eta)\,\mathrm{d}\eta \int_0^H \mu(z)\psi'(z, k_j')\chi_q(z)\,\mathrm{d}z. \tag{74}$$

It is clear from (74) that

$$I_{pq} = I_{qp}. \tag{75}$$

Expressions for I_{pq} are evaluated in the Appendix.

The coefficients D_{ip} and D_{jq} are determined subject to the assumption that $F(\tau_i, \tau_j)$ in (72) is stationary with respect to variations in the D_{ip}'s, which implies

$$\partial F/\partial D_{ip} = 0, \qquad p = 1, 2, \ldots, r \tag{76}$$

and

$$\partial F/\partial D_{jq} = 0, \qquad q = 1, 2, \ldots, r. \tag{77}$$

We thus obtain a set of r linear, algebraic equations for D_{ip}, $p = 1, 2, \ldots, r$ and another set of r linear, algebraic equations for D_{jq}, $q = 1, 2, \ldots, r$. If we solve the two sets of equations for D_{ip} and D_{jq}, $p, q = 1, 2, \ldots, r$ and substitute their values in (72), we obtain the required entry s_{ij} of the scattering matrix.

Following the procedure outlined above, we obtain the following scattering matrix in the special case $r=1$, $s \geqslant 1$ (when there is a single (fundamental) mode in the left-hand domain and there are s modes on the right):

$$
\boldsymbol{S} = \frac{1}{I_{11}} \begin{pmatrix} 1 & \lambda_{11}P_{11} & \lambda_{21}P_{21} & \cdots & \lambda_{s1}P_{s1} \\ \lambda_{11}P_{11} & \lambda_{11}^2 P_{11}^2 & \lambda_{11}P_{11}\lambda_{21}P_{21} & \cdots & \lambda_{11}P_{11}\lambda_{s1}P_{s1} \\ \lambda_{21}P_{21} & \lambda_{11}P_{11}\lambda_{21}P_{21} & \lambda_{21}^2 P_{21}^2 & \cdots & \lambda_{21}P_{21}\lambda_{s1}P_{s1} \\ \vdots & & & \ddots & \\ \lambda_{s1}P_{s1} & \lambda_{11}P_{11}\lambda_{s1}P_{s1} & \cdots & \cdots & \lambda_{s1}^2 P_{s1}^2 \end{pmatrix},
$$

$$
\boldsymbol{K} \pm \mathrm{i}\boldsymbol{S} = \frac{k_1}{I'_{11}} \begin{pmatrix} I'_{11} \pm \mathrm{i} & \pm \mathrm{i}\lambda_{11}P_{11} & \pm \mathrm{i}\lambda_{21}P_{21} & \cdots & \pm \mathrm{i}\lambda_{s1}P_{s1} \\ \pm \mathrm{i}\lambda_{11}P_{11} & \lambda_{11}^2(I'_{11} + \mathrm{i}P_{11}^2) & \pm \mathrm{i}\lambda_{11}P_{11}\lambda_{21}P_{21} & \cdots & \pm \mathrm{i}\lambda_{11}P_{11}\lambda_{s1}P_{s1} \\ \vdots & & \lambda_{21}^2(I'_{11} \pm \mathrm{i}P_{21}^2) & \cdots & \cdots \\ \vdots & & & \ddots & \\ \pm \mathrm{i}\lambda_{s1}P_{s1} & \pm \mathrm{i}\lambda_{11}P_{11}\lambda_{s1}P_{s1} & \cdots & \cdots & \lambda_{s1}^2(I'_{11} \pm \mathrm{i}P_{s1}^2) \end{pmatrix},
$$

$$
I'_{11} = k_1 I_{11},
$$

$$
(\boldsymbol{K} + \mathrm{i}\boldsymbol{S})^{-1} = \frac{1}{k_1(N - \mathrm{i}I'_{11})}
$$

$$
\times \begin{pmatrix} (N-1) - \mathrm{i}I'_{11} & -P_{11}/\lambda_{11} & -P_{21}/\lambda_{21} & \cdots & -P_{s1}/\lambda_{s1} \\ -P_{11}/\lambda_{11} & \lambda_{1}^{2}[(N - P_{11}^2) - \mathrm{i}I'_{11}] & -2P_{11}P_{21}/\lambda_{11}\lambda_{21} & \cdots & -P_{11}P_{s1}/\lambda_{s1}\lambda_{11} \\ -P_{21}/\lambda_{21} & -P_{11}P_{21}/\lambda_{11}\lambda_{21} & \lambda_{21}^{-2}[(N - P_{21}^2) - \mathrm{i}I'_{11}] & \cdots & -P_{21}P_{s1}/\lambda_{21}\lambda_{s1} \\ \vdots & & & \ddots & \\ -P_{s1}/\lambda_{s1} & -P_{s1}P_{11}/\lambda_{s1}\lambda_{11} & \cdots & \cdots & \lambda_{s1}^{-2}[(N - P_{s1}^2) - \mathrm{i}I'_{11}] \end{pmatrix},
$$

where $N = 1 + P_{11}^2 + P_{21}^2 + \ldots + P_{s1}^2$,

$$
\boldsymbol{T} = (\boldsymbol{K} + \mathrm{i}\boldsymbol{S})^{-1}(\boldsymbol{K} - \mathrm{i}\boldsymbol{S}) = \frac{1}{N - \mathrm{i}I'_{11}}
$$

$$
\times \begin{pmatrix} (N-2) - \mathrm{i}I'_{11} & -2\lambda_{11}P_{11} & -2\lambda_{21}P_{21} & \cdots & -2\lambda_{s1}P_{s1} \\ -2P_{11}/\lambda_{11} & (N - 2P_{11}^2) - \mathrm{i}I'_{11} & -2P_{11}\lambda_{21}P_{21}/\lambda_{11} & \cdots & -2P_{11}\lambda_{s1}P_{s1}/\lambda_{11} \\ -2P_{21}/\lambda_{21} & -2P_{21}P_{11}\lambda_{11}/\lambda_{21} & (N - 2P_{21}^2) - \mathrm{i}I'_{11} & \cdots & -2P_{21}\lambda_{s1}P_{s1}/\lambda_{21} \\ \vdots & & & & \\ -2P_{s1}/\lambda_{s1} & -2P_{s1}P_{11}\lambda_{11}/\lambda_{s1} & \cdots & \cdots & (N - 2P_{s1}^2) - \mathrm{i}I'_{11} \end{pmatrix}. \tag{78}
$$

If we set $I'_{11} = 0$ in (78), we obtain the matrix given by equation (62) for the corresponding case in the plane-wave approximation. Hence (78) is a variational improvement of the matrix obtained in the plane-wave approximation, and the effects of non-propagated modes are incorporated in the parameter I'_{11}. We may remark that the total incident energy averaged over time is exactly partitioned into the time average of the total reflected and total transmitted energies concentrated in normal modes, and that no energy is carried by non-propagated modes.

Acknowledgments

The author wishes to thank Dr J A Hudson for many valuable and helpful discussions and suggestions, and also the British Council for the award of a TATD fellowship. This work was carried out at the Department of Applied Mathematics and Theoretical Physics at Cambridge University, UK.

Appendix

Consider

$$
\begin{aligned}
I(m) &= \int_0^H \mu(z)\chi_m(z)\psi'(z, k_j')\,\mathrm{d}z \\
&= \int_0^{h_1} \mu_1\phi_1^{(m)}(z)\psi_1'(z, k_j')\,\mathrm{d}z + \int_{h_1}^{H} \mu_2\phi_2^{(m)}(z)\psi_2'(z, k_j')\,\mathrm{d}z \\
&= I_1 + I_2 \qquad \text{(say)},
\end{aligned}
\tag{A1}
$$

where $\phi_1^{(m)}(z)$ and $\phi_2^{(m)}(z)$ are given by (8) and (9), and $\psi_1'(z, k_j')$ and $\psi_2'(z, k_j')$ are given by (22).

$$
\begin{aligned}
I_1 &= \frac{\mu_1 D_m D(\lambda_j')}{\cos(\sigma_1^{(m)}h_1)\cos(\sigma_1'(\lambda_j')h_2)} \int_0^{h_1} \cos(\sigma_1^{(m)}z)\cos[(z+\delta)\sigma_1'(\lambda_j')]\,\mathrm{d}z, \qquad \lambda_j' = -k_j'^2 \\
&= \frac{\mu_1 D_m D(\lambda_j')}{[\sigma_1'(\lambda_j')]^2 - [\sigma_1^{(m)}]^2}[\sigma_1'(\lambda_j')\tan(h_2\sigma_1'(\lambda_j')) - \sigma_1^{(m)}\tan\sigma_1^{(m)}h_1] \\
&\quad - \frac{\mu_1 D_m(\lambda_j')\sin(\sigma_1'(\lambda_j)\delta)}{\cos\sigma_1^{(m)}h_1\cos(\sigma_1'(\lambda_j)h_2)[(\sigma_1'(\lambda_j'))^2 - (\sigma_1^{(m)})^2]} \\
&= \frac{D_m D(\lambda_j')}{(\sigma_1'(\lambda_j'))^2 - (\sigma_1^{(m)})^2}\{\mu_2\sigma_2'(\lambda_j')\tanh[\sigma_2'(\lambda_j')(H-h_1)] \\
&\quad - \mu_2\sigma_2^{(m)}\tanh[\sigma_2^{(m)}(H-h_1)]\} \\
&\quad - \frac{\mu_1 D_m D(\lambda_j')\sin[\sigma_1'(\lambda_j)\delta]}{\cos\sigma_1^{(m)}h_1\cos[\sigma_1'(\lambda_j)h_2][(\sigma_1'(\lambda_j'))^2 - (\sigma_1^{(m)})^2]}
\end{aligned}
\tag{A2}
$$

(using dispersion equations (16) and (20), and

$$
\begin{aligned}
I_2 &= \frac{\mu_2 D_m D(\lambda_j')}{\cosh[\sigma_2^{(m)}(H-h_1)]\cosh[\sigma_2'(\lambda_j')(H-h_1)]} \\
&\quad \times \int_{h_1}^{H} \cosh[\sigma_2^{(m)}(z-H)]\cosh[\sigma_2'(\lambda_j')(z-H)]\,\mathrm{d}z \\
&= \frac{\mu_2 D_m D(\lambda_j')}{[\sigma_2'(\lambda_j')]^2 - [\sigma_2^{(m)}]^2}\{\sigma_2'(\lambda_j')\tanh[\sigma_2'(\lambda_j')(H-h_1)] \\
&\quad - \sigma_2^{(m)}\tanh[\sigma_2^{(m)}(H-h_1)]\},
\end{aligned}
\tag{A3}
$$

whence from (A1), (A2) and (A3) we obtain

$$I(m)=\frac{-\mu_1 D_m(\lambda_j')}{k_j'^2+k_m^2}\,\frac{\sin[\sigma_1'(\lambda_j)\delta]}{\cos[\sigma_1^{(m)}h_1]\cos[\sigma_1'(\lambda_j)h_2]}. \tag{A4}$$

From (74) and (A4) we have the following representation of I_{pq} as an infinite series which can be shown to converge in k' (well enough for numerical work):

$$I_{pq}=\sum_{j=1}^{\infty}\frac{I(p)I(q)}{k_j'} \tag{A5}$$

(with $I(p)$, $I(q)$ given by equation (A4)).

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